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Non-Markovianity in collision models with initial intra-environment correlations

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E-mail: gpleasance1@gmail.com, aeneira@mun.ca, merkli@mun.ca and francesco.petruccione@nithecs.ac.za**Keywords:** collision models, quantum non-markovianity, entanglement

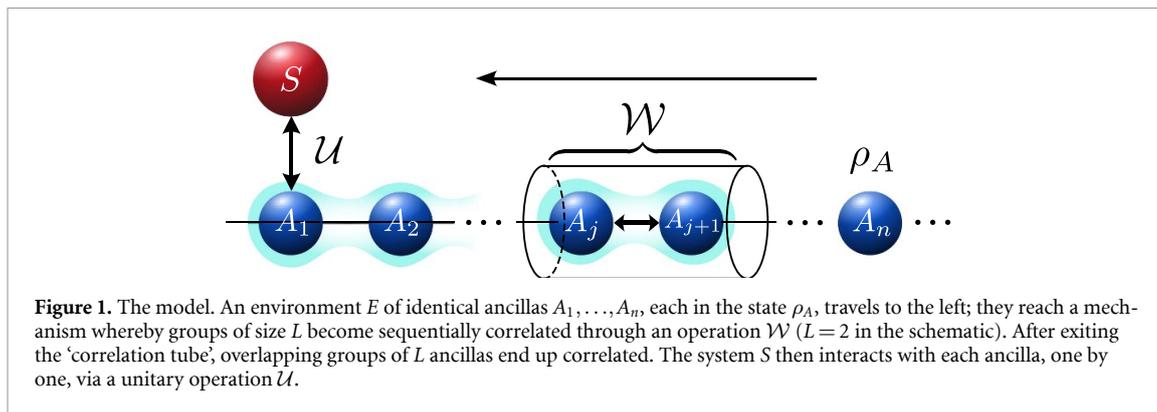
Abstract

Collision models (CMs) describe an open system interacting in sequence with elements of an environment, termed ancillas. They have been established as a useful tool for analyzing non-Markovian open quantum dynamics based on the ability to control the environmental memory through simple feedback mechanisms. In this work, we investigate how ancilla–ancilla (AA) entanglement can serve as a mechanism for controlling the non-Markovianity of an open system, focusing on an operational approach to generating correlations within the environment. To this end, we first demonstrate that the open dynamics of CMs with sequentially generated correlations between groups of ancillas can be mapped onto a composite CM, where the memory part of the environment is incorporated into an enlarged Markovian system. We then apply this framework to an all-qubit CM, and show that non-Markovian behavior emerges only when the next incoming pair of ancillas are entangled prior to colliding with the system. On the other hand, when system-ancilla collisions precede AA entanglement, we find the open dynamics to always be Markovian. Our findings highlight how certain qualitative features of inter-ancilla correlations can strongly influence the onset of system non-Markovianity.

1. Introduction

Open quantum systems are fundamental to a wide range of quantum technologies—such as quantum sensors and quantum thermal devices [1, 2]—where the behavior of the system of interest inherently depends on its interaction with the environment [3]. Under a standard framework in which the environment is assumed to be memoryless and weakly interact with the system, the resulting dynamics may be well approximated by a GKSL master equation describing a Markovian evolution [4, 5]. Such master equations have the general advantage of being amenable to both analytical and numerical techniques [6] as well as offering a physically transparent interpretation of the open dynamics [7–9]. Nonetheless, the growing importance of non-Markovian quantum dynamics [10] has led to the development of alternative frameworks for treating open system dynamics with memory, including the hierarchical equations of motion [11, 12], the pseudomode method [13–17], path integral techniques [18], and stochastic approaches [19–21].

Among these techniques, collision models (CMs) have emerged as a promising tool for studying open quantum systems in a way that realizes the system-reservoir interaction in a conceptually simple manner [22, 23]. Rather than assuming the system to interact continuously with the same environment, CMs divide up the environment into a chain of discrete ancillas, which each interact with the system via step-wise collisions. Since this setup offers both a high degree of control and flexibility over the type of evolution it describes [24–26], such models have found a broad range of applications spanning quantum optics [27–29] and quantum information [30, 31], to quantum thermodynamics [32–35] and



high energy physics [36]. Along similar lines, CMs are well suited to simulating non-Markovian dynamics due to their capability in realizing memory effects through simple feedback mechanisms [24, 25, 28, 29, 34, 37–51]. These mechanisms may be categorized according to various schemes that rely on modifying the basic CM used for Markovian dynamics—either by allowing multiple system-ancilla collisions to occur at each step [28, 37, 48], or by changing the properties of the environment. Under the latter approach, memory effects may be introduced via environments featuring internal ancilla–ancilla (AA) collisions [38–40, 46, 47, 50], or with initially correlated states [24, 25, 42–44]. Across all such cases, the precise mechanism underpinning the memory is the establishment of correlations between the system and a specific ‘interacting’ portion of the environment. Indeed in [40, 46], the repeated erasure of correlations between the system and the last collided ancilla, which then undergoes AA interactions, was generally shown to alter the non-Markovianity of the open dynamics. On the other hand, ancillas which the system has already collided with, but that have no involvement in later interactions (the ‘non-interacting’ portion of the environment), may be completely neglected from the open dynamics despite possibly retaining correlations with the system.

Based on this link between system-environment correlations and memory effects, Campbell *et al* [46] recently introduced the notion of ‘memory depth’ for non-Markovian CMs with AA collisions that enacts a partitioning of the environment into its memory and non-memory parts. The memory depth quantifies the size of the memory in terms of the interacting portion of ancillas that become correlated with the system following consecutive AA collisions—i.e. those whose correlations cannot be neglected without generally changing the open dynamics—and hence must be accounted for in the construction of the system dynamical map. As per [46], this suggests an effective Markovian description of the open dynamics in which the system and memory are treated as a composite system interacting with the remaining non-memory part [45]. Conceptually, such a description is related to the idea of non-Markovian to Markovian mappings employed more widely in the open quantum systems literature [13–16].

In this work, we devise a similar strategy to [46] for treating CMs with correlated environment states, which is used to analyze the role inter-ancilla entanglement plays in the non-Markovianity of an all-qubit CM. To do so, we introduce a class of non-Markovian CMs whereby overlapping groups of consecutive ancillas are correlated in sequence *before* colliding with the system; see figure 1. This differs from the scheme of [46] where SA and AA collisions occur in alternating sequence starting with initially uncorrelated ancillas. At the same time, our approach bridges the gap between CMs with correlated environments [24, 25, 42–44] and their experimental implementation [29, 52, 53], in that our scheme accounts for the physical preparation of the initial environment state. We demonstrate that this setup allows for an effective Markovian description of the open dynamics in terms of a composite CM (CCM) akin to [45, 46], thereby reflecting an analogous partitioning of the environment into memory and non-memory parts. However, contrary to [46], the construction of the dynamical map is shown to differ not only on the order in which AA and SA collisions are applied, but also the portion of environmental ancillas that the system effectively interacts with.

For an all-qubit CM with sequentially correlated AA pairs, it is further established that entanglement between ancillas within the interacting portion of the environment is *necessary* for the emergence of non-Markovian behavior. Moreover, we show that non-Markovianity appears precisely because such ancilla pairs are entangled before colliding with the system. Indeed, when the ancillas are instead correlated after the system collides with the first ancilla, then the dynamics is entirely Markovian, regardless of the degree of AA entanglement.

The remainder of this paper is organized as follows. In section 2, we introduce the CM setup displayed in figure 1, where intra-environment correlations are generated via a family of unitary maps applied to consecutive groups of ancillas. In section 3 we demonstrate how the open dynamics of this setup can be mapped onto a composite CM, which incorporates the memory part of the environment into an enlarged Markovian open system. We then in section 4 apply our result to an all-qubit CM and study the connection between the degree of AA entanglement of relevant ancilla pairs and the non-Markovianity of the open dynamics. Finally, conclusions and outlook are presented in section 5.

2. Basic CM

To fix the notation, we start by outlining the basic setup used to describe an open system S interacting with a memoryless environment of uncorrelated ancillas [23]. In such a model the environment E consists of n identical ancillas A_j ($j = 1, \dots, n$) initialized in the state $\rho_E(0) = \rho_{A_1} \otimes \dots \otimes \rho_{A_n}$, where the Hilbert spaces of the system and the j th ancilla are denoted as \mathcal{H}_S and \mathcal{H}_{A_j} , respectively. Since the ancillas are identical, the index j is simply a label for a copy of the same Hilbert space \mathcal{H}_A . The system and environment interact through pairwise collisions between S and each A_j , where each collision is described by a unitary operator U acting on $\mathcal{H}_S \otimes \mathcal{H}_A$. We shall denote by U_{S_j} the equivalent actions on S and the ancilla A_j when several ancillas are present, i.e. when the total Hilbert space is $\mathcal{H}_S \otimes \mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n}$, and

$$\rho \mapsto \mathcal{U}_{S_j} \rho = U_{S_j} \rho U_{S_j}^\dagger \quad (1)$$

the collision map between S and A_j acting on density matrices ρ of the system plus n ancillas.

Let us define the single-collision map

$$\rho_S \mapsto \Lambda[\rho_S] = \text{Tr}_A [U(\rho_S \otimes \rho_A) U^\dagger]. \quad (2)$$

It is a completely positive and trace-preserving (CPTP) map on the set of all system density matrices. For an initially factorized state $\rho(0) = \rho_S(0) \otimes \rho_A^{\otimes n}$, the reduced system state after n collisions $\rho_S(n) = \text{Tr}_{A_1 \dots A_n} [\mathcal{U}_{S_n} \dots \mathcal{U}_{S_1} \rho(0)]$ is obtained by iterating n times the map Λ ,

$$\rho_S(n) = \Lambda^n[\rho_S(0)]. \quad (3)$$

This implies the system dynamics to be Markovian insofar that the composite map $\Lambda_n \equiv \Lambda^n$ implementing the evolution $\rho_S(0) \mapsto \rho_S(n) = \Lambda_n \rho_S(0)$ is divisible, i.e. Λ_n can inherently be written as a composition of CPTP maps [54, 55]. At the same time, the memoryless property of equation (3) may also be understood from fact that it describes the repeated interaction between S and a *single* ancilla A , whose correlations with S can be erased at each step without affecting the future system evolution (that is, no memory of the system state is retained across successive collisions).

3. Non-Markovian CM with correlated ancillas

As previously noted, the realization of non-Markovian dynamics relies on modifying the basic setup above according to one of several schemes. Two common schemes employed in the literature are:

- (i) AA collisions: After S has collided with the ancilla A_j , and before S collides with the next one A_{j+1} , the two ancillas A_j and A_{j+1} interact via a unitary operator $W_{j+1,j}$ [34, 38–40, 46, 47, 49]. As such, A_{j+1} already carries memory of the system state before colliding with S . In this scheme one can also collide groups of more than two ancillas; the total ‘length’ of each group (number of connections between first and last ancilla) is called the memory depth [46].
- (ii) Intra-environment correlations: The initial environment state $\rho_E(0)$ is replaced by one that is nonfactorizable, reflecting classical or quantum correlations between ancillas [24, 25, 42, 44].

In the current work, we introduce a different approach to the latter scheme (ii) that closely resembles (i), in which correlations between ancillas are generated operationally. In particular, we consider initial states in which overlapping groups of L ancillas are correlated in sequence before colliding with S (see figure 1). We proceed to formalize the model as follows. First, let W define a unitary acting on any fixed number of $L \geq 1$ ancillas, and let $W_{[j+L-1,j]}$ denote W acting on a string of L ancillas with indices $j, j+$

$1, \dots, j+L-1$ (while acting as the identity on all other ancillas). We further denote by $\mathcal{W}_{[j+L-1,j]}$ and \mathcal{W} the actions of $W_{[j+L-1,j]}$ and W lifted to density matrices of S and n ancillas,

$$\mathcal{W}\rho = W\rho W^\dagger, \quad \mathcal{W}_{[L+j-1,j]}\rho = W_{[L+j-1,j]}\rho W_{[L+j-1,j]}^\dagger. \quad (4)$$

Thus, the initial states we consider are given as

$$\rho(0) = \rho_S(0) \otimes \mathcal{W}_{[n+L-1,n]} \cdots \mathcal{W}_{[L+1,2]} \mathcal{W}_{[L,1]} (\rho_A^{\otimes n+L-1}). \quad (5)$$

By setting $L=1$, it is clear that equation (5) reduces to the same class of product states associated with memoryless CMs. In this regard, we refer to L as the *correlation length* of the environment in analogy to the memory depth introduced in [46] (see further discussion below). We remark that it is also feasible to allow for more general types of operations \mathcal{W} represented by non-unitary CPTP maps.

Starting from the initial condition (5), the full system-reservoir state evolves as

$$\begin{aligned} \rho(n) &= \mathcal{U}_{S_n} \cdots \mathcal{U}_{S_1} (\rho_S(0) \otimes \mathcal{W}_{[n+L-1,n]} \cdots \mathcal{W}_{[L,1]} \rho_A^{\otimes n+L-1}) \\ &= \mathcal{U}_{S_n} \mathcal{W}_{[n+L-1,n]} \cdots \mathcal{U}_{S_1} \mathcal{W}_{[L,1]} (\rho_S(0) \otimes \rho_A^{\otimes n+L-1}), \end{aligned} \quad (6)$$

such that the reduced system state after n collisions reads

$$\begin{aligned} \rho_S(n) &= \text{Tr}_{A_1 \dots A_{n+L-1}} [\mathcal{U}_{S_n} \mathcal{W}_{[n+L-1,n]} \cdots \mathcal{U}_{S_1} \mathcal{W}_{[L,1]} (\rho_S(0) \otimes \rho_A^{\otimes n+L-1})] \\ &\equiv \Lambda_n [\rho_S(0)]. \end{aligned} \quad (7)$$

In contrast to equation (3), the map Λ_n is generally non-Markovian since it is not guaranteed to be factorizable into products of CPTP maps.

Remark. The operators U and W commute exactly when \mathcal{U} and \mathcal{W} do. In this case, we have

$$\mathcal{U}_{S_n} \cdots \mathcal{U}_{S_1} \mathcal{W}_{[n+L-1,n]} \cdots \mathcal{W}_{[L,1]} = \mathcal{W}_{[n+L-1,n]} \cdots \mathcal{W}_{[L,1]} \mathcal{U}_{S_n} \cdots \mathcal{U}_{S_1}, \quad [U, W] = 0. \quad (8)$$

When tracing over the ancilla degrees of freedom in equation (7), the action of each \mathcal{W} now disappears since these superoperators represent a conjugation with a unitary. It then follows that the reduced system dynamics is independent of the intra-environment correlations provided these are generated by an operator W that commutes with the $S-A$ interaction.

3.1. Mapping onto Markovian CCM

Following the ideas of [45, 46], we now demonstrate how the system dynamics Λ_n can be realized as a Markovian process of an enlarged system containing S and $L-1$ ancillas, an approach known as Markovian embedding. The main advantage of such an embedding is that the dynamics of the enlarged system may be treated using techniques applicable to maps with the same composite structure as equation (3). On the other hand, the dimension of this system grows exponentially with the correlation length L ; for example, if the ancillas are of the dimension $\dim(\mathcal{H}_A) = d$, then the dimension of the enlarged system is $\dim(\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L-1}) = \dim(\mathcal{H}_S) \cdot d^{L-1}$. Hence, such a mapping in practice is limited to small values of L .

Our first result is that the dynamical map Λ_n , equation (7), may be expressed as the n th iteration of another map \mathcal{M} that transforms between states of S and *different* groups of $L-1$ consecutive ancillas, as depicted schematically in figure 2(a) (below the index of the ancillas is suppressed since the state of each ancilla is represented on a duplicated Hilbert space \mathcal{H}_A). To state this formally, we introduce $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L-1})$ as the space of all density matrices of S plus $L-1$ ancillas, and denote by ϑ an arbitrary element of $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L-1})$. We then have the following result.

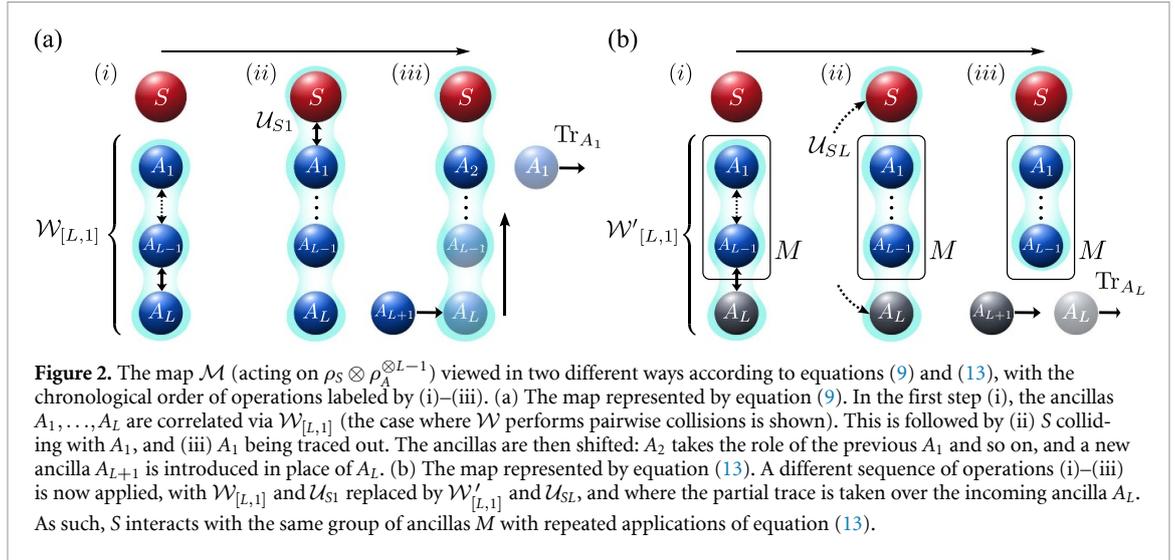
Theorem 1. Let $\mathcal{M} : \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L-1}) \rightarrow \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L-1})$ be the following CPTP map,

$$\vartheta \mapsto \mathcal{M}[\vartheta] = \text{Tr}_{A_1} (\mathcal{U}_{S_1} \mathcal{W}_{[L,1]} (\vartheta \otimes \rho_A)). \quad (9)$$

Then the reduced system state (7) after n collisions is expressed as

$$\rho_S(n) = \text{Tr}_{A_{L-1} \dots A_1} \mathcal{M}^n [\rho_S(0) \otimes \rho_A^{\otimes L-1}], \quad (10)$$

where \mathcal{M}^n is the n th power of \mathcal{M} , and the trace in equation (10) is taken over all $L-1$ ancillas.



We present a proof of Theorem 1 in section 6.1. To clarify the interpretation of equation (10), consider the $L=2$ case where AA pairs are sequentially correlated. In this scenario, the map equation (9) describes a sequence of three operations involving the system S and two ancillas, say A_1 and A_2 . These ancillas are first correlated through the operation $\mathcal{W}_{[2,1]}$. This is followed by a collision between S and A_1 , and finally the ancilla A_1 is traced out. The resulting $\mathcal{M}[\vartheta]$ now represents the state of S plus ancilla A_2 . Applying the same map again then takes the state of the enlarged system $S + A_2$, and implements the same three operations to obtain the state of $S + A_3$. Continuing this process n times we obtain $\mathcal{M}^n[\vartheta]$, i.e. the state of S and ancilla A_{n+1} , so that tracing over A_{n+1} recovers the reduced system state from equation (7). We also highlight that the map \mathcal{M} , equation (9) itself depends on the ancilla state ρ_A , which is considered arbitrary but fixed.

Composite system representation of the open dynamics. We next prove that the description of the dynamics given by equation (10) coincides with an alternative representation whereby correlations are generated between A_{j+L-1} and the *same* group of $L-1$ ancillas $M = \{A_1, \dots, A_{L-1}\}$, rather than between a different subset of ancillas at each iteration. To this end, let $\rho \mapsto \mathcal{S}_{k,l}\rho = \mathcal{S}_{k,l}\rho\mathcal{S}_{k,l}$ define the swap map that interchanges the states of the ancillas with indices k and l , according to (omitting the tensor product symbols)

$$\mathcal{S}_{k,l}[\rho_{A_1} \cdots \rho_{A_k} \cdots \rho_{A_l} \cdots \rho_{A_n}] = \rho_{A_1} \cdots \rho_{A_l} \cdots \rho_{A_k} \cdots \rho_{A_n}. \quad (11)$$

We then define the operator

$$\mathcal{W}'_{[L,1]} = \mathcal{S}_{L,L-1} \cdots \mathcal{S}_{2,1} \mathcal{W}_{[L,1]} \quad (12)$$

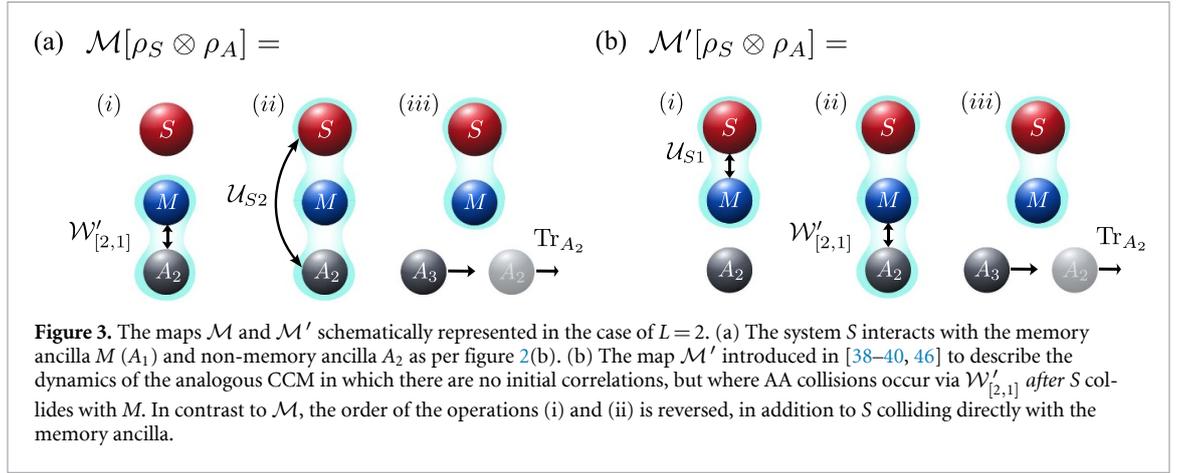
and show in section 6.2 the following result.

Proposition 1. *The map \mathcal{M} defined in equation (9) may alternatively be written in the form*

$$\mathcal{M}[\vartheta] = \text{Tr}_{A_L} \left(\mathcal{U}_{SL} \mathcal{W}'_{[L,1]} (\vartheta \otimes \rho_A) \right), \quad (13)$$

where the trace is taken over the ancilla A_L .

The interpretation of the system dynamics as viewed by equation (13) is that an incoming ancilla A_L is initially correlated with a collection M of $L-1$ ancillas via $\mathcal{W}'_{[L,1]}$, which then collides with S , before being traced out and replaced by another ancilla A_{L+1} ; see figure 2(b). Notably, because it is the same incoming ancilla that is traced out after each SA collision, the ancillas M can be interpreted as the *memory part* of the environment whose correlations with the system generally play a nontrivial role in the system evolution [46]. On the other hand, the ancillas not contained in this group represent the non-memory part insofar that their correlations with S can be erased at each step without affecting the system dynamics. We recall that a similar interpretation was applied to the map Λ defined in the basic CM, equation (3). Our result—that the generally non-Markovian system dynamics Λ_n can be obtained as the reduction of the Markovian dynamics \mathcal{M} of a larger (system-ancilla) complex—establishes what is



referred to in the literature as a Markovian embedding, or equivalently a non-Markovian to Markovian mapping of the open dynamics [13, 15, 16]. We emphasize that such a mapping holds for arbitrary values of the parameter L , in addition to being valid for arbitrary operations \mathcal{U} and \mathcal{W} represented by CPTP maps.

Naturally, since the correlation length quantifies the size of the memory part retaining essential correlations with the system, it adopts the same role as the memory depth from [46] for CMs whose non-Markovianity stems from AA collisions. This in turn establishes a direct correspondence between the two mechanisms (i) and (ii), in which the correlations forming the memory are either dynamically generated through AA collisions [40, 46], or through intra-environment correlations present in the initial state (with a specific correlation structure determined by equation (5)). It should be noted, however, that the role of the memory in these two schemes differs based on the structure of the CCMs governing the respective system dynamics. In particular, the map \mathcal{M}' describing the evolution of the composite system $S + A_1$ for a memory depth $L=2$ is given as⁵ [46]

$$\vartheta \mapsto \mathcal{M}'[\vartheta] = \text{Tr}_{A_2} \left(\mathcal{W}'_{[2,1]} \mathcal{U}_{S1} (\vartheta \otimes \rho_A) \right). \quad (14)$$

According to this map, the ancilla A_1 first collides with S , which then collides with the non-memory part of the environment, as shown in figure 3(b). On the other hand, for our setup with $L=2$, shown in figure 3(a), the order of these operations is *reversed*, in addition to the fact that the system only collides with the memory A_1 through the non-memory part of the environment. We demonstrate below how these differences in the ordering of SA and AA collisions can significantly impact the non-Markovianity of the dynamics.

4. All-qubit CM

In this section, we present our main results on the role of AA entanglement on the non-Markovianity of an open two-level system. For simplicity, we focus on the case where the ancillas A_j are also two-level systems, i.e. $\mathcal{H}_S = \mathcal{H}_{A_j} = \mathbb{C}^2$. We further assume a correlation length of $L=2$ and take the SA and AA collision operators to be

$$U_{Sj} = e^{-i\tau\sigma_x \otimes \sigma_x^{(j)}}, \quad W_{[j+1,j]} = e^{-i\varepsilon\sigma_z^{(j+1)} \otimes \sigma_z^{(j)}}. \quad (15)$$

Here, $\sigma_{x,y,z}$ ($\sigma_{x,y,z}^{(j)}$) are the Pauli spin- $\frac{1}{2}$ operators of the system (j th ancilla), and $\tau, \varepsilon \geq 0$ define the interaction phase and entanglement phase, respectively.

4.1. Reduced system dynamics and decoherence function

The explicit form of the map \mathcal{M} , equation (9), is given by

$$\mathcal{M}[\vartheta] = \text{Tr}_{A_1} \left(e^{-i\tau\sigma_x \otimes \sigma_x^{(1)} \otimes \mathbb{1}} e^{-i\varepsilon\mathbb{1} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(1)}} (\vartheta \otimes \rho_A) e^{i\varepsilon\mathbb{1} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(1)}} e^{i\tau\sigma_x \otimes \sigma_x^{(1)} \otimes \mathbb{1}} \right), \quad (16)$$

⁵ To maintain consistency with our definition of L , the map equation (14) corresponds to a memory depth of $L=2$, whereas in [46] it corresponds to a memory depth of one. In both definitions the memory is represented by the same ancilla A_1 .

where ϑ is the state of S and a given ancilla A_1 . Hence, by defining $\vartheta_n = \mathcal{M}^n[\rho_S(0) \otimes \rho_A]$, the reduced system state (10) takes the form

$$\rho_S(n) = \text{Tr}_{A_1, A_2} \left(e^{-i\tau\sigma_x \otimes \sigma_x^{(1)} \otimes \mathbb{1}} e^{-i\varepsilon\mathbb{1} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(1)}} (\vartheta_{n-1} \otimes \rho_A) e^{i\varepsilon\mathbb{1} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(1)}} e^{i\tau\sigma_x \otimes \sigma_x^{(1)} \otimes \mathbb{1}} \right). \quad (17)$$

Let $|\pm\rangle$ ($|\pm\rangle_A$) denote the eigenstates of σ_x ($\sigma_x^{(j)}$). Using equation (17), it is straightforward to verify that the populations $\langle \pm | \rho_S(n) | \pm \rangle$ are invariant in the basis $\{|+\rangle, |-\rangle\}$, i.e. $\langle \pm | \rho_S(n) | \pm \rangle = \langle \pm | \rho_S(n-1) | \pm \rangle$, and for $\langle + | \rho_S(n) | - \rangle$ to be proportional to $\langle + | \rho_S(n-1) | - \rangle$. Hence, we may write

$$\rho_S(n) = \begin{pmatrix} p & D(n)z \\ D(n)^* z^* & 1-p \end{pmatrix}, \quad (\sigma_x\text{-basis}) \quad (18)$$

where $D(n) \in \mathbb{C}$ is the decoherence function of S , $D(0) = 1$, and $p = \langle + | \rho_S(0) | + \rangle$, $z = \langle + | \rho_S(0) | - \rangle$.

By employing a Liouville space representation of the map equation (16), the decoherence function $D(n)$ can be computed as a sum of matrix elements of \mathcal{M}^n , which in general depends on all parameters n , ε , τ , and the initial state ρ_A . To show this, we parametrize ρ_A by its matrix elements in the σ_z basis as

$$\rho_A = \frac{1}{2} \begin{pmatrix} 1 + \rho_3 & \rho_1 + \rho_2 \\ \rho_1 - \rho_2 & 1 - \rho_3 \end{pmatrix}, \quad (\sigma_z\text{-basis}), \quad (19)$$

where $\rho_1^2 + |\rho_2|^2 + \rho_3^2 \leq 1$. We then have the following result, proved in section 6.3.

Proposition 2. *The decoherence function $D(n)$ in equation (18) is given by*

$$D(n) = \sum_{m=0}^3 [(\cos(2\tau)\mathbf{A}_1 - i\sin(2\tau)\mathbf{A}_2)_{0m}^n] \rho_m, \quad (20)$$

where the ρ_m , $m = 0, \dots, 3$ are given by equation (19) and $[\dots]_{kl}$ denotes the k, l matrix element of the matrices

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho_1 \cos(2\varepsilon) & 0 & 0 & -i\rho_2 \sin(2\varepsilon) \\ \rho_2 \cos(2\varepsilon) & 0 & 0 & -i\rho_1 \sin(2\varepsilon) \\ \rho_3 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & \cos(2\varepsilon) & -i\rho_3 \sin(2\varepsilon) & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & \rho_3 \cos(2\varepsilon) & -i\sin(2\varepsilon) & 0 \end{pmatrix}. \quad (21)$$

For simplicity, we will assume hereon for each ancilla to be initialized in the state

$$\rho_A = |+\rangle\langle +|_A, \quad (22)$$

corresponding to $\rho_0 = \rho_1 = 1$, and $\rho_2 = \rho_3 = 0$.

4.2. Effect of AA entanglement on the system non-Markovianity

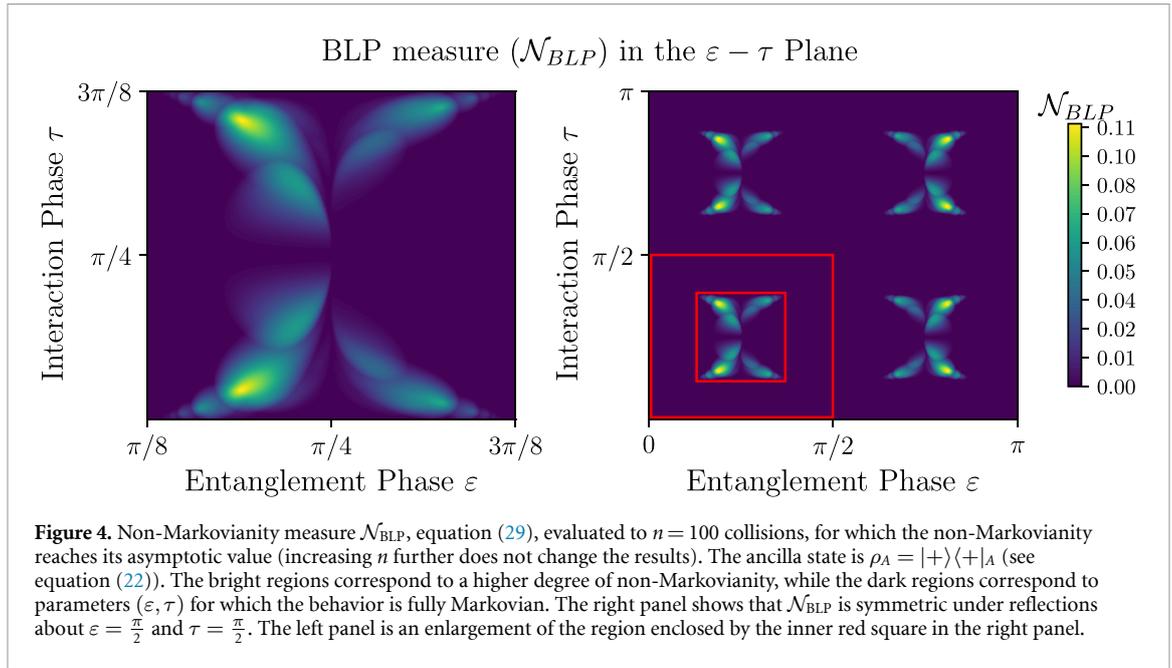
Measure of non-Markovianity. We now introduce a suitable quantifier of non-Markovianity based on the trace distance measure proposed in [56]. The trace distance quantifies the distinguishability of any pair of density matrices ρ^1 and ρ^2 , and is defined as

$$\mathcal{D}(\rho^1, \rho^2) = \frac{1}{2} \|\rho^1 - \rho^2\|_1, \quad (23)$$

where $\|\cdot\|_1$ is the trace norm $\|X\|_1 = \text{Tr}\sqrt{X^\dagger X}$. The states ρ^1 and ρ^2 are maximally distinguishable when their trace distance equals one, which occurs if and only if ρ^1 and ρ^2 are orthogonal. In a Markovian process, the distinguishability between any pair of initial system states $\rho_S^{1,2}(0)$ only ever decreases with increasing collision number n . This specifically amounts to a continual loss of information from the system to the environment. On the other hand, non-Markovian effects are identified when information flows from the environment back to the system, corresponding to a temporary increase in the trace distance between successive collisions.

For the discrete evolution $n \mapsto \rho_S(n)$, this behavior is captured by the the Breuer-Piilo-Laine (BLP) measure [56, 57]

$$\mathcal{N}_{\text{BLP}} = \max_{\rho_S^1(0), \rho_S^2(0)} \sum_{n \in \mathcal{S}_+} [\mathcal{D}(\rho_S^1(n+1), \rho_S^2(n+1)) - \mathcal{D}(\rho_S^1(n), \rho_S^2(n))], \quad (24)$$



which by definition is maximized over every choice of initial state pair $\{\rho_S^{1,2}(0)\}$. The sum is taken over the set

$$S_+ = \{n \in \mathbb{N} : \mathcal{D}(\rho_S^1(n+1), \rho_S^2(n+1)) - \mathcal{D}(\rho_S^1(n), \rho_S^2(n)) > 0\}. \quad (25)$$

As such, equation (24) is non-zero if and only if there is some initial state pair for which the trace distance temporarily increases. For the all-qubit CM considered above, it is possible to evaluate the trace distance as

$$\mathcal{D}(\rho_S^1(n), \rho_S^2(n)) = \sqrt{(p_1 - p_2)^2 + |D(n)|^2 |z_1 - z_2|^2}, \quad (26)$$

implying the set S_+ to be independent of the initial states $\rho_S^{1,2}(0)$, i.e.

$$n \in S_+ \iff |D(n+1)| > |D(n)|. \quad (27)$$

Hence, both a necessary and sufficient condition for non-Markovianity in this case is for $|D(n)|$ to increase when going from collision n to $n+1$.

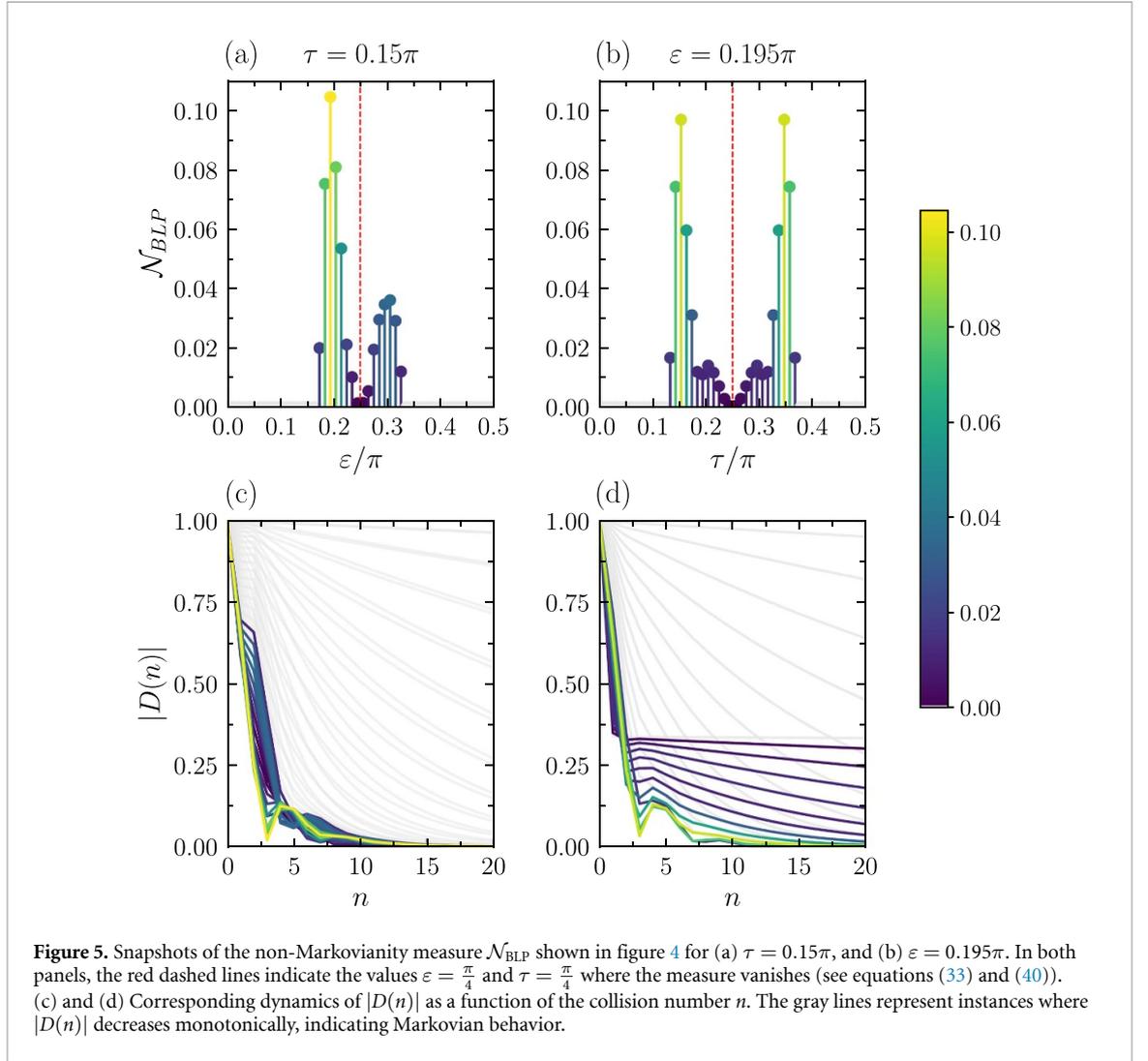
For qubits, the maximal value in equation (24) is achieved for pairs of states that are antipodal points on the Bloch sphere [58]. The trace distance between any pair of antipodal states can be parametrized in terms of a single parameter $\eta \in [0, 1]$,

$$\mathcal{D}(\rho_S^1(n), \rho_S^2(n)) = \sqrt{\eta + |D(n)|^2(1-\eta)}, \quad \rho_S^1 \perp \rho_S^2. \quad (28)$$

The maximization in equation (24) is then a maximization over η :

$$\mathcal{N}_{BLP} = \max_{\eta \in [0,1]} \sum_{n \in S_+} \left(\sqrt{\eta + |D(n+1)|^2(1-\eta)} - \sqrt{\eta + |D(n)|^2(1-\eta)} \right). \quad (29)$$

Numerical results. In figure 4, we display the measure \mathcal{N}_{BLP} as a function of (ε, τ) for a total of $n = 100$ collisions (capping S_+ at $n = 100$). The lighter areas indicate parts of the parameter space where memory effects play a non-trivial role in the reduced system dynamics, while darker areas represent instances when the dynamics is Markovian (or approximately so). The decoherence function $D(n)$ has symmetries which are inherited by \mathcal{N}_{BLP} and it suffices to consider the parameter range $(\varepsilon, \tau) = [\frac{\pi}{8}, \frac{3\pi}{8}] \times [\frac{\pi}{8}, \frac{3\pi}{8}]$ depicted in the left panel of figure 4, since the behavior outside this region can be reconstructed by reflections. Furthermore, the non-Markovianity vanishes within the region enclosed between the red lines in the right panel of the same figure; this occurs trivially on the boundaries where ε and τ are either 0 or $\frac{\pi}{2}$. In particular, for $\varepsilon = 0$ and $\varepsilon = \frac{\pi}{2}$, no AA correlations are generated and the CM reduces



to the basic setup where the system map Λ_n has a semigroup structure (see equation (3)). It is found that the non-Markovianity is approximately maximal for

$$\varepsilon_{\max} \approx 0.195\pi, \quad \tau_{\max} \approx 0.15\pi, 0.35\pi, \quad (30)$$

for which there are two global maxima (in the considered parameter range) with respect to τ . Additional snapshots of the non-Markovianity measure are presented in figure 5 for fixed values of the entanglement and interaction phases. Here, the corresponding dynamics of $|D(n)|$ is also shown (see equation (28)).

Comparing now the AA entanglement to the degree of non-Markovianity of the dynamics, we choose to quantify the entanglement between any consecutive pair of ancillas $A_n - A_{n+1}$ immediately *before* A_n collides with S via the concurrence [59],

$$\mathcal{C}(\rho_{A_n A_{n+1}}) = \max\{0, \lambda_1(n) - \lambda_2(n) - \lambda_3(n) - \lambda_4(n)\}. \quad (31)$$

Here, $\lambda_1(n) \geq \lambda_2(n) \geq \lambda_3(n) \geq \lambda_4(n)$ are the square roots of the eigenvalues of the matrix $\rho_{A_n A_{n+1}}(\sigma_y^{(n)} \otimes \sigma_y^{(n+1)})\rho_{A_n A_{n+1}}^*(\sigma_y^{(n)} \otimes \sigma_y^{(n+1)})$, and $\rho_{A_n A_{n+1}}^*$ is the complex conjugate of $\rho_{A_n A_{n+1}}$ (when expressed as a matrix in the $\sigma_z \otimes \sigma_z$ basis). The density matrix $\rho_{A_n A_{n+1}}$ representing the interacting portion of the environment (i.e. the subset of ancillas that are correlated with S before their interaction) before the n th collision may be obtained as

$$\begin{aligned} \rho_{A_n A_{n+1}} &= \text{Tr}_{S A_1 \dots A_{n-1}} [\mathcal{U}_{S n-1} \dots \mathcal{U}_{S 1} \mathcal{W}_{[n+1, n]} \dots \mathcal{W}_{[2, 1]} (\rho_S(0) \otimes \rho_A^{\otimes n+1})] \\ &= \text{Tr}_{A_1 \dots A_{n-1}} [\mathcal{W}_{[n+1, n]} \dots \mathcal{W}_{[2, 1]} (\rho_A \otimes \dots \otimes \rho_A)] \\ &= \mathcal{W}_{[n+1, n]} (\text{Tr}_{A_{n-1}} [\mathcal{W}_{[n, n-1]} \dots \text{Tr}_{A_1} [\mathcal{W}_{[2, 1]} (\rho_A \otimes \rho_A)] \dots \otimes \rho_A] \otimes \rho_A). \end{aligned} \quad (32)$$

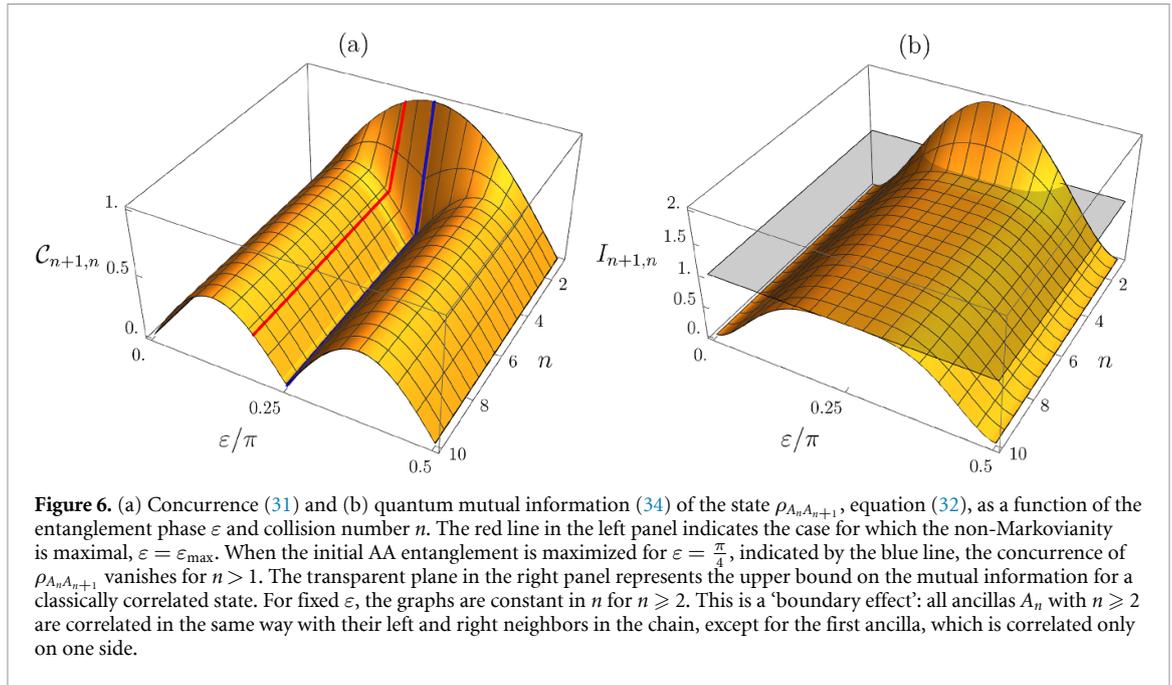


Figure 6. (a) Concurrence (31) and (b) quantum mutual information (34) of the state $\rho_{A_n A_{n+1}}$, equation (32), as a function of the entanglement phase ε and collision number n . The red line in the left panel indicates the case for which the non-Markovianity is maximal, $\varepsilon = \varepsilon_{\max}$. When the initial AA entanglement is maximized for $\varepsilon = \frac{\pi}{4}$, indicated by the blue line, the concurrence of $\rho_{A_n A_{n+1}}$ vanishes for $n > 1$. The transparent plane in the right panel represents the upper bound on the mutual information for a classically correlated state. For fixed ε , the graphs are constant in n for $n \geq 2$. This is a ‘boundary effect’: all ancillas A_n with $n \geq 2$ are correlated in the same way with their left and right neighbors in the chain, except for the first ancilla, which is correlated only on one side.

Note that the second equality follows from the cyclicity of the trace. Hence, since the state $\rho_{A_n A_{n+1}}$ is independent of the $S - A$ interaction, the concurrence (31) depends only on the initial preparation of the environment through \mathcal{W} . In fact, equation (32) is obtained equivalently by taking the trace over all ancillas except A_n and A_{n+1} in the initial state (5).

In figure 6, we display the behavior of the concurrence with varying entanglement phase ε after $n - 1$ collisions (starting immediately before S collides with A_1). Interestingly, we see that $\mathcal{C}(\rho_{A_n A_{n+1}})$ is symmetric about $\varepsilon = \frac{\pi}{4}$ and remains constant for all AA pairs beyond $n = 1$. This is in contrast to the non-Markovianity measure in figure 5(a) which is asymmetric over the same interval. In the case of $\varepsilon = \frac{\pi}{4}$, highlighted by the blue line in figure 6(a), the concurrence is maximal for the first AA pair but vanishes for all subsequent pairs. The decoherence function (20) for this value of ε can be evaluated as (see section 6.4)

$$\lim_{\varepsilon \rightarrow \pi/4} D(n) = (\cos(2\tau))^n, \quad (33)$$

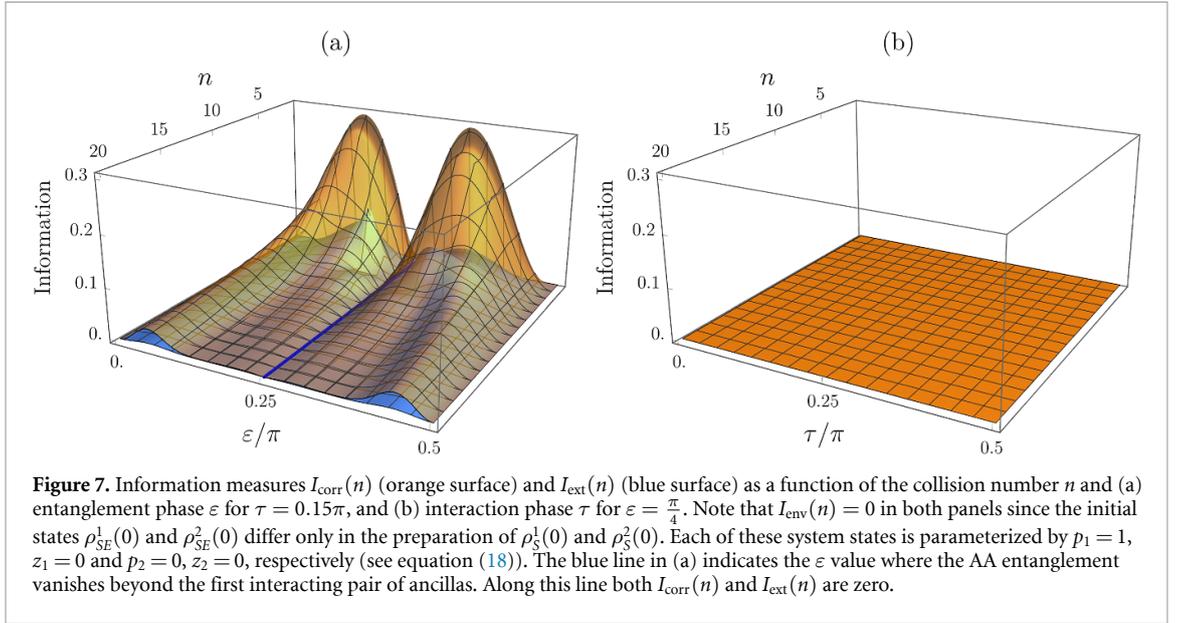
which is strictly nonincreasing in n for all τ . Thus, all system non-Markovianity is *eliminated* when the concurrence (31) of the state $\rho_{A_n A_{n+1}}$ vanishes. We emphasize that the Markovian behavior in this case is qualitatively different to the basic CM where each $A_n - A_{n+1}$ pair are uncorrelated before A_n collides with S . To verify this, we examine the quantum mutual information [60] of the state (32) in figure 6(b),

$$I(\rho_{A_n A_{n+1}}) = S(\rho_{A_n}) + S(\rho_{A_{n+1}}) - S(\rho_{A_n A_{n+1}}), \quad (34)$$

where $S(\rho) = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy of ρ . The quantum mutual information $I(\rho_{A_n A_{n+1}})$ provides a measure of the total correlations between A_n and A_{n+1} and is zero only if the two ancillas are in a product state. Apart from the cases $\varepsilon = 0$ and $\varepsilon = \frac{\pi}{2}$ corresponding to the basic Markovian setup, we find that ancillas $A_n - A_{n+1}$ are generally correlated for all values of the entanglement phase. In particular, classical correlations between A_n and A_{n+1} are maximized when $\varepsilon = \frac{\pi}{4}$ and $n > 1$. This indicates how the transition from non-Markovian to Markovian behavior precisely coincides with the elimination of entanglement between ancillas A_n and A_{n+1} beyond the first interacting pair: as such, we conclude that, within the interacting portion of the environment, AA entanglement is *necessary* for the dynamics to be non-Markovian. Moreover, the elimination of AA entanglement resulting from the saturation of entanglement between the first two ancillas may be interpreted as an effect of entanglement monogamy [61, 62].

To illustrate why non-entangled AA pairs lead to Markovian behavior, let us consider the following upper bound on the trace distance variation between collisions n and m ($m \geq n$) [63]

$$\mathcal{D}(\rho_S^1(m), \rho_S^2(m)) - \mathcal{D}(\rho_S^1(n), \rho_S^2(n)) \leq I_{S A_n}(n), \quad (35)$$



with

$$I_{SA_n}(n) = \mathcal{D}(\rho_{SA_n}^1(n), \rho_S^1(n) \otimes \rho_{A_n}^1(n)) + \mathcal{D}(\rho_{SA_n}^2(n), \rho_S^2(n) \otimes \rho_{A_n}^2(n)) + \mathcal{D}(\rho_{A_n}^1(n), \rho_{A_n}^2(n)). \quad (36)$$

Here, $\rho_{SA_n}(n)$ represents the joint state of $S - A_n$ immediately *before* the n th collision (i.e. after the application of $\mathcal{W}_{[n+1, n]}$), while $\rho_S(n) = \text{Tr}_{A_n}[\rho_{SA_n}(n)]$ and $\rho_{A_n}(n) = \text{Tr}_S[\rho_{SA_n}(n)]$ represent the reduced states of S and the ancilla A_n (as before, the superscripts denote different initial preparations of the system S). The upper bound $I_{SA_n}(n)$ is composed of three terms measuring, respectively, the total correlations between S and A_n , and the distinguishability of the two reduced states $\rho_{A_n}^1$ and $\rho_{A_n}^2$. The inequality (35) implies that a nonmonotonic increase in the trace distance can therefore only occur if the system and the interacting portion of ancillas are correlated before the previous n th collision, and/or if the corresponding reduced ancilla states are different. Importantly, this highlights how the emergence of memory effects and information backflow is causally related to the development of correlations between S and A_n . Based on this analysis, we may introduce the quantities

$$I_{\text{corr}}(n) = \mathcal{D}(\rho_{SA_n}^1(n), \rho_S^1(n) \otimes \rho_{A_n}^1(n)) + \mathcal{D}(\rho_{SA_n}^2(n), \rho_S^2(n) \otimes \rho_{A_n}^2(n)), \quad (37)$$

$$I_{\text{env}}(n) = \mathcal{D}(\rho_{A_n}^1(n), \rho_{A_n}^2(n)), \quad (38)$$

which measure information contained within system-environment correlations and through the distinguishability of pairs of reduced ancilla states, respectively. We also introduce the measure

$$I_{\text{ext}}(n) = \mathcal{D}(\rho_{SA_n}^1(n), \rho_{SA_n}^2(n)) - \mathcal{D}(\rho_S^1(n), \rho_S^2(n)) \quad (39)$$

representing the external information held by ancilla A_n [64].

In figure 7, we display the behavior of these information measures with varying entanglement and interaction phases. Figure 7(a) illustrates the case where the interaction phase is set to $\tau = \tau_{\text{max}}$, corresponding to when the non-Markovianity reaches its maximum value for $\varepsilon = \varepsilon_{\text{max}}$. Here, the correlation measure $I_{\text{corr}}(n)$ is found to be non-zero for almost all values of ε , while $I_{\text{env}}(n)$ vanishes due to the initial condition $\rho_E^1(0) = \rho_E^2(0)$ (recall that E labels the total environment of $n + 1$ ancillas). The non-Markovianity of the dynamics is thus seen to stem from the generation of correlations between S and A_n , since the only contribution to the upper bound equation (36) in this case is from $I_{\text{corr}}(n)$. The interpretation of non-Markovianity resulting from $S - A_n$ correlations is also consistent with the fact that both $I_{\text{corr}}(n)$ and $I_{\text{ext}}(n)$ vanish along the line $\varepsilon = \frac{\pi}{4}$, coinciding with the loss of AA entanglement and the transition from non-Markovian to Markovian dynamics (see figures 5(a) and 6(a)). In particular, the same quantities are shown in figure 7(b) with varying τ and fixed entanglement phase $\varepsilon = \frac{\pi}{4}$, where it is found that all three measures vanish. This demonstrates how the transition to Markovian behavior is caused by the loss of correlations between the system S and ancilla A_n , which occurs in parallel to the loss of AA entanglement between the interacting portion of ancillas.

Finally, we note that for $\tau = \frac{\pi}{4}$, the decoherence function $D(n)$, equation (20) satisfies the following limit (proven in section 6.4):

$$\lim_{\tau \rightarrow \pi/4} D(n) = e^{-i\pi n/2} \cos(2\varepsilon), \quad n \geq 1. \quad (40)$$

The system behavior is therefore Markovian in this case regardless of the value of ε , given that $|D(n)|$ reaches its steady state value after the first collision. This is consistent with the numerical results shown in figures 4 and 5(b), in which \mathcal{N}_{BLP} vanishes for $\tau = \frac{\pi}{4}$.

4.3. Interchanging the order of SA and AA collisions

As we have seen in the previous section, AA entanglement is necessary for the system evolution (17) to exhibit memory effects. We further show this occurrence of non-Markovianity is due to the fact that the ancillas are entangled *before* they collide with the system.

For this, we proceed to evaluate the dynamical map (16) in the opposite case, where consecutive ancillas are entangled *after* the first member in each pair collides with S . In particular, this coincides with the CCM representation of the open dynamics depicted in figure 3(b). The dynamical map in this case reads

$$\mathcal{M}'[\varrho] = \text{Tr}_{A_1} \left(e^{-i\varepsilon \mathbb{1} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(1)}} \left(\left[e^{-i\tau \sigma_x \otimes \sigma_x^{(1)}} \varrho e^{i\tau \sigma_x \otimes \sigma_x^{(1)}} \right] \otimes \rho_A \right) e^{i\varepsilon \mathbb{1} \otimes \sigma_z^{(2)} \otimes \sigma_z^{(1)}} \right). \quad (41)$$

We show in section 6.5 that the corresponding decoherence function is given by

$$D'(n) = e^{-2i\tau} (\cos(2\tau) - i \sin(2\tau) \cos(2\varepsilon))^{n-1}, \quad n \geq 1. \quad (42)$$

Hence, $|D'(n)| = (\cos^2(2\tau) + \sin^2(2\tau) \cos^2(2\varepsilon))^{\frac{1}{2}(n-1)}$, which is a function of n that never increases, since $\cos^2(2\tau) + \sin^2(2\tau) \cos^2(2\varepsilon) \leq 1$. We conclude that the system dynamics is Markovian for all values of the parameters τ and ε . This illustrates how reversing the order of SA and AA operations can profoundly affect the non-Markovianity of the system evolution.

5. Conclusions

In summary, we have derived a new class of non-Markovian CMs with correlated environment states, whose correlations are generated through applying a sequence of unitary (or CPTP) operations to successive groups of ancillas. For these types of CMs, we have demonstrated that the open dynamics admits a general mapping onto a CCM, in which the memory part of the environment is incorporated into an enlarged Markovian system. In particular, the size of this memory part is quantified by the correlation length characterizing the range of intra-environment correlations between ancillas. This shares an analogous interpretation with the memory depth from [46], which, for a separate class of CMs featuring AA collisions, quantifies the range of system-environment correlations that must be accounted for in the construction of the system dynamical map. Notably, the CCM corresponding to our setup displays key differences to that of [46] regarding the arrangement of the memory and non-memory parts and their mutual interactions with the system.

We have further analyzed the extent to which entanglement can influence the non-Markovianity of the open dynamics of an all-qubit CM with correlated ancilla pairs. By examining the nature of correlations shared between ancilla pairs representing the interacting portion of the environment, we have found AA entanglement to be necessary (although not sufficient) for the emergence of non-Markovian behavior. At the same time, the order in which SA and AA collisions occur has been shown to play a key role in determining the non-Markovianity of the open dynamics. A more detailed analysis of the role of AA entanglement in the emergence of non-Markovianity will be presented elsewhere [65].

Looking ahead, the CM developed here could be used to study the thermodynamics of open systems interacting with nonthermal environments, whereby ancillas initially prepared in thermal states are correlated in sequence. Understanding how such correlations impact the performance of thermal devices—taking into account the energetic cost in generating such correlations—may have applications in the design of quantum heat engines and refrigerators [66]. On the experimental side, our model could be implemented via the same platforms used to realize CMs featuring AA collisions, including NMR [52] and optical setups [29, 53], or it could also be simulated on a quantum computer [67]. Finally, one could also extend our work in the direction of [24, 25, 45] by formally characterizing the non-Markovian dynamics simulable by a CM with a correlation structure determined by equation (5). For an open two-level system, it has been demonstrated that any CP- or P-indivisible Pauli channel can be simulated by a CM with suitably correlated ancillas [24, 25]. While the precise connection between

non-Markovianity and intra-environment correlations is not yet fully understood, such work could shed further light on the applicability of CMs for simulating general non-Markovian open dynamics.

6. Proof of results

6.1. Proof of theorem 1

Starting from equation (7),

$$\rho_S(n) = \text{Tr}_{A_{n+L-1}, \dots, A_1} \left[\text{Tr}_{A_n} \mathcal{U}_{S_n} \mathcal{W}_{[n, n-L+1]} \cdots \text{Tr}_{A_1} \mathcal{U}_{S_1} \mathcal{W}_{[L, 1]} (\rho_S \otimes \rho_A^{\otimes n+L-1}) \right], \quad (43)$$

we define the map $\widetilde{\mathcal{M}} : \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L}) \rightarrow \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L})$ acting on density matrices ϱ of S and L ancillas by

$$\widetilde{\mathcal{M}}[\varrho] = (\text{Tr}_{A_1} \mathcal{U}_{S_1} \mathcal{W}_{[L, 1]} \varrho) \otimes \rho_A. \quad (44)$$

Here, the trace is taken over the first ancilla space (the one that S interacts with via \mathcal{U}_{S_1}). The operator $\widetilde{\mathcal{M}}[\varrho]$ is again a density matrix of the system plus L ancillas, with the first ancilla traced out and replaced by a fresh one in the state ρ_A . Writing $\widetilde{\mathcal{M}}^n = \widetilde{\mathcal{M}} \circ \cdots \circ \widetilde{\mathcal{M}}$, we obtain from equation (43),

$$\rho_S(n) = \text{Tr}_{A_L, \dots, A_1} \widetilde{\mathcal{M}}^n [\rho_S \otimes \rho_A^{\otimes L}]. \quad (45)$$

Now let ϑ_0 be a density matrix of the system S plus $L-1$ ancillas. According to equation (44), we have $\widetilde{\mathcal{M}}[\vartheta_0 \otimes \rho_A] = \vartheta_1 \otimes \rho_A$, where ϑ_1 is again a density matrix of S plus $L-1$ ancillas. The definition of the map \mathcal{M} from equation (9) implies $\mathcal{M}[\vartheta_0] = \vartheta_1$, so that $\widetilde{\mathcal{M}}[\vartheta_0 \otimes \rho_A] = \mathcal{M}[\vartheta_0] \otimes \rho_A$, and

$$\widetilde{\mathcal{M}}^n [\vartheta_0 \otimes \rho_A] = \mathcal{M}^n [\vartheta_0] \otimes \rho_A. \quad (46)$$

Combining this with equation (45) finally provides equation (10) of the main text.

It is straightforward to further prove the CPTP property of \mathcal{M} by noting that

$$\mathcal{M}[\vartheta] = \text{Tr}_{A_L} (\mathcal{S}_{L,1} \mathcal{U}_{S_1} \mathcal{W}_{[L,1]} (\vartheta \otimes \rho_A)) = \text{Tr}_{A_L} (U(\vartheta \otimes \rho_A) U^\dagger).$$

Here, $\mathcal{S}_{L,1}$ is a swap map that switches the first and last ancillas in the tensor product $\mathcal{H}_S \otimes \mathcal{H}_A^{\otimes L}$, and U is a fixed unitary operator. It directly follows from Stinespring's theorem [68] that the resulting map $\vartheta \mapsto \text{Tr}_{A_L} (U(\vartheta \otimes \rho_A) U^\dagger)$ is CPTP. This completes the proof of Theorem 1. \square

6.2. Proof of proposition 1

We first note that the swap map $\mathcal{S}_{k,l}$ acting on any ancilla pair A_k and A_l obeys the identities

$$\mathcal{S}_{k,l}^2 = \mathcal{I}, \quad \mathcal{U}_{S_1} \mathcal{S}_{2,1} \mathcal{S}_{3,2} \cdots \mathcal{S}_{L,L-1} = \mathcal{S}_{2,1} \mathcal{S}_{3,2} \cdots \mathcal{S}_{L,L-1} \mathcal{U}_{S_L}, \quad (47)$$

where \mathcal{I} denotes the identity map on $\mathcal{H}_A \otimes \mathcal{H}_A$. The second identity relates to the fact that transferring the state of A_L to A_1 , colliding S with A_1 , and then shifting the state of A_1 back to A_L , is in effect the same as colliding S with A_L . Using the explicit form of \mathcal{M} as per equation (9),

$$\mathcal{M}[\vartheta] = \text{Tr}_{A_1} (\mathcal{U}_{S_1} \mathcal{W}_{[L,1]} (\vartheta \otimes \rho_A)),$$

we apply equation (47) to obtain

$$\begin{aligned} \mathcal{M}[\vartheta] &= \text{Tr}_{A_1} (\mathcal{U}_{S_1} \mathcal{S}_{2,1} \mathcal{S}_{3,2} \cdots \mathcal{S}_{L,L-1} \mathcal{S}_{L,L-1} \cdots \mathcal{S}_{3,2} \mathcal{S}_{2,1} \mathcal{W}_{[L,1]} (\vartheta \otimes \rho_A)) \\ &= \text{Tr}_{A_1} (\mathcal{S}_{2,1} \mathcal{S}_{3,2} \cdots \mathcal{S}_{L,L-1} \mathcal{U}_{S_L} \mathcal{W}'_{[L,1]} (\vartheta \otimes \rho_A)), \end{aligned} \quad (48)$$

with $\mathcal{W}'_{[L,1]} = \mathcal{S}_{L,L-1} \cdots \mathcal{S}_{2,1} \mathcal{W}_{[L,1]}$. Next we use that

$$\text{Tr}_{A_k} (\mathcal{S}_{k+1,k} \rho_S \rho_{A_1} \cdots \rho_{A_k} \rho_{A_{k+1}} \cdots) = \rho_S \cdots \rho_{A_k} \rho_{A_{k+2}} \cdots = \text{Tr}_{A_{k+1}} (\rho_S \cdots \rho_{A_k} \rho_{A_{k+1}} \cdots). \quad (49)$$

By linearity and applying this relation $L-1$ times to the last line of equation (48), we then get equation (13). This completes the proof of proposition 1. \square

6.3. Proof of proposition 2

For the model outlined in section 4, in which both S and each A are qubits, we may express \mathcal{M} in the following ordered operator basis

$$\{e_{00}, e_{01}, e_{02}, e_{03}, \quad e_{10}, e_{11}, e_{12}, e_{13}, \quad e_{20}, e_{21}, e_{22}, e_{23}, \quad e_{30}, e_{31}, e_{32}, e_{33}\}, \quad (50)$$

where

$$e_{kl} = e_k \otimes e_l, \quad (51)$$

and

$$e_0 = \frac{1}{\sqrt{2}}, \quad e_1 = \frac{1}{\sqrt{2}}\sigma_x, \quad e_2 = -\frac{i}{\sqrt{2}}\sigma_y, \quad e_3 = \frac{1}{\sqrt{2}}\sigma_z. \quad (52)$$

(To simplify the notation we shall use the same e_k to denote basis operators of either S or A .) The operator sets $\{e_j\}$ are mutually orthonormal with respect to the Hilbert–Schmidt inner product

$$\langle e_k | e_l \rangle \equiv \text{Tr} [e_k^\dagger e_l] = \delta_{k,l}, \quad (53)$$

such that

$$\langle e_{kl} | e_{mn} \rangle = \langle e_k \otimes e_l | e_m \otimes e_n \rangle = \langle e_k | e_m \rangle \langle e_l | e_n \rangle = \delta_{k,m} \delta_{l,n}. \quad (54)$$

Accordingly, when expressed in the basis equation (50), the map \mathcal{M} takes the form of a 16×16 matrix with matrix elements

$$\mathbf{M}_{kl, mn} \equiv \langle e_{kl} | \mathbf{M} | e_{lm} \rangle = \text{Tr} [(e_k \otimes e_l)^\dagger \mathcal{M} [e_m \otimes e_n]],$$

while in the same basis, the composite state $\vartheta \mapsto |\vartheta\rangle$ of $S + A$ is represented as 16×1 column vector. The coefficients ρ_m parameterizing ρ_A may also be computed from the inner product equation (53) as

$$\rho_m = \sqrt{2} \langle e_m | \rho_A \rangle = \sqrt{2} \text{Tr} [e_m^\dagger \rho_A], \quad (55)$$

with $\rho_0 = 1$.

After much algebra, the matrix \mathbf{M} can be shown to take the form

$$\mathbf{M} = \left(\begin{array}{cc|cc} \mathbf{A}_1 & 0 & & \\ 0 & \mathbf{A}_1 & & \\ \hline & & \cos(2\tau)\mathbf{A}_1 & i\sin(2\tau)\mathbf{A}_2 \\ & & i\sin(2\tau)\mathbf{A}_2 & \cos(2\tau)\mathbf{A}_1 \end{array} \right). \quad (56)$$

Here, the empty blocks have null entries, and the matrices \mathbf{A}_1 and \mathbf{A}_2 are given by

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho_1 \cos(2\varepsilon) & 0 & 0 & -i\rho_2 \sin(2\varepsilon) \\ \rho_2 \cos(2\varepsilon) & 0 & 0 & -i\rho_1 \sin(2\varepsilon) \\ \rho_3 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & \cos(2\varepsilon) & -i\rho_3 \sin(2\varepsilon) & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & \rho_3 \cos(2\varepsilon) & -i\sin(2\varepsilon) & 0 \end{pmatrix}. \quad (57)$$

The structure of equation (56) into the 4×4 blocks corresponds to the four groupings of the ordered basis indicated by a space in equation (50).

The matrix \mathbf{A}_1 has a simple eigenvalue 1, and only one other eigenvalue 0 of algebraic multiplicity 3 and geometric multiplicity 2, implying it is nondiagonalizable. However, powers of \mathbf{A}_1 are straightforward to calculate,

$$\mathbf{A}_1^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho_1 \cos(2\varepsilon) - i\rho_2 \rho_3 \sin(2\varepsilon) & 0 & 0 & 0 \\ \rho_2 \cos(2\varepsilon) - i\rho_1 \rho_3 \sin(2\varepsilon) & 0 & 0 & 0 \\ \rho_3 & 0 & 0 & 0 \end{pmatrix} \quad n \geq 2. \quad (58)$$

Hence, the n th power of equation (56) is then

$$\mathbf{M}^n = \left(\begin{array}{cc|cc} \mathbf{A}_1^2 & 0 & & \\ 0 & \mathbf{A}_1^2 & & \\ \hline & & \mathbf{B}^n & \end{array} \right), \quad n \geq 2, \quad (59)$$

where the 8×8 matrix \mathbf{B} is defined as

$$\mathbf{B} = \left(\begin{array}{c|c} \cos(2\tau)\mathbf{A}_1 & i \sin(2\tau)\mathbf{A}_2 \\ \hline i \sin(2\tau)\mathbf{A}_2 & \cos(2\tau)\mathbf{A}_1 \end{array} \right). \tag{60}$$

Using the above expression for \mathbf{M}^n , we next show that the decoherence function $D(n)$ can be evaluated in terms of powers of the matrices $\mathbf{A}_{1,2}$, equation (20).

Lemma 1. *The decoherence function $D(n)$ in equation (18) is given by*

$$D(n) = \sum_{m=0}^3 \left([\mathbf{M}^n]_{30,3m} - [\mathbf{M}^n]_{20,3m} \right) \rho_m. \tag{61}$$

Proof. In the representation introduced above, we can evaluate the reduced density matrix $\rho_S(n) = \text{Tr}_A \mathcal{M}^n[\rho_S \otimes \rho_A]$ by acting \mathbf{M}^n on the column vector $|\rho_S \otimes \rho_A\rangle$. Since $\text{Tr}_A[e_{kl}] = \sqrt{2}e_k\delta_{l,0}$, the parts of the image of \mathbf{M}^n proportional to e_{kl} all vanish except if $l=0$, and hence we do not need all matrix elements of \mathbf{M}^n to determine $\rho_S(n)$. Explicitly, we have

$$\begin{aligned} \rho_S(n) &= \sum_{k,l=0}^3 \text{Tr}_A[e_{kl}] \langle e_{kl} | \mathbf{M}^n | \rho_S \otimes \rho_A \rangle \\ &= \sqrt{2} \sum_{k=0}^3 e_k \langle e_{k0} | \mathbf{M}^n | \rho_S \otimes \rho_A \rangle \\ &= \sqrt{2} \sum_{k=0}^3 e_k \sum_{l,m=0}^3 [\mathbf{M}^n]_{k0,lm} \langle e_{lm} | \rho_S \otimes \rho_A \rangle. \end{aligned} \tag{62}$$

Next we have from equation (55),

$$\langle e_{lm} | \rho_S \otimes \rho_A \rangle = \frac{1}{\sqrt{2}} \rho_m \text{Tr} \left[e_l^\dagger \rho_S \right], \quad l, m = 0, \dots, 3,$$

which combined with equation (62) yields

$$\rho_S(n) = \sum_{k=0}^3 e_k \sum_{l,m=0}^3 [\mathbf{M}^n]_{k0,lm} \rho_m \text{Tr} \left[e_l^\dagger \rho_S \right]. \tag{63}$$

We can further simplify this expression using (see equations (59) and (56))

$$[\mathbf{M}^n]_{00,lm} = \delta_{l,0} \delta_{m,0}, \quad [\mathbf{M}^n]_{10,lm} = \delta_{l,1} \delta_{m,0}, \tag{64}$$

so that

$$\rho_S(n) = \frac{1}{\sqrt{2}} e_0 + e_1 \text{Tr} [e_1 \rho_S] + \sum_{k=2,3} e_k \sum_{l,m=0}^3 [\mathbf{M}^n]_{k0,lm} \rho_m \text{Tr} \left[e_l^\dagger \rho_S \right]. \tag{65}$$

The initial state ρ_S represented in the σ_x basis reads

$$\rho_S = \begin{pmatrix} p & z \\ z^* & 1-p \end{pmatrix}, \quad (\sigma_x\text{-basis}) \tag{66}$$

with $p \in [0, 1]$ and $|z|^2 \leq p(1-p)$. Let us then also represent the reduced density matrix equation (65) in the same basis,

$$\rho_S(n) = \begin{pmatrix} p & \alpha_3(n) - \alpha_2(n) \\ \alpha_3(n) + \alpha_2(n) & 1-p \end{pmatrix}, \quad (\sigma_x\text{-basis}) \tag{67}$$

where

$$\alpha_k(n) = \frac{1}{\sqrt{2}} \sum_{l,m=0}^3 [\mathbf{M}^n]_{k0,lm} \rho_m \text{Tr} \left[e_l^\dagger \rho_S \right], \quad k = 2, 3. \tag{68}$$

In terms of the parameters p and z , we have

$$\text{Tr} \left[e_l^\dagger \rho_S \right] = \sqrt{2} \begin{cases} \frac{1}{2}, & l = 0 \\ p - \frac{1}{2}, & l = 1 \\ -i \text{Im} z, & l = 2 \\ \text{Re} z, & l = 3 \end{cases} \quad (69)$$

The reduced system dynamics is thus entirely determined by the off-diagonal matrix elements of equation (67) depending on $\alpha_2(n)$ and $\alpha_3(n)$ (in the σ_x basis). We now define

$$\alpha_3(n) - \alpha_2(n) = D(n)z, \quad (70)$$

where $D(n)$ is a function of n which is independent of z . In particular, both off-diagonal elements of $\rho_S(n)$ must vanish for all n if $z = 0$. It then follows that $\alpha_2(n) = \alpha_3(n) = 0$ if $z = 0$. From equations (56) and (69), the terms with $l = 0, 1$ in equation (68) can also be seen to vanish, and so

$$\alpha_k(n) = \frac{1}{\sqrt{2}} \sum_{l=2,3} \sum_{m=0}^3 [\mathbf{M}^n]_{k0,lm} \rho_m \text{Tr} \left[e_l^\dagger \rho_S \right], \quad k = 2, 3. \quad (71)$$

Using again equation (69) we obtain

$$\alpha_k(n) = \left(\sum_{m=0}^3 [\mathbf{M}^n]_{k0,3m} \rho_m \right) \text{Re} z - i \left(\sum_{m=0}^3 [\mathbf{M}^n]_{k0,2m} \rho_m \right) \text{Im} z, \quad k = 2, 3. \quad (72)$$

For the left side of equation (70) to be a multiple of z , we have the constraint

$$\sum_{m=0}^3 \left([\mathbf{M}^n]_{30,3m} - [\mathbf{M}^n]_{20,3m} \right) \rho_m = - \sum_{m=0}^3 \left([\mathbf{M}^n]_{30,2m} - [\mathbf{M}^n]_{20,2m} \right) \rho_m \quad (73)$$

and the decoherence function $D(n)$ becomes equation (61). This completes the proof of Lemma 1. \square

Next we evaluate the matrix elements appearing in the sum equation (61). We first obtain from equation (56),

$$D(n) = \sum_{m=0}^3 \left([\mathbf{B}^n]_{4,m+4} - [\mathbf{B}^n]_{0,m+4} \right) \rho_m, \quad (74)$$

where \mathbf{B} is given in equation (60). We observe that \mathbf{B} is a block circulant matrix and that it can be block-diagonalized according to

$$\mathbf{B} = \frac{1}{2} \begin{pmatrix} \mathbb{1}_4 & \mathbb{1}_4 \\ \mathbb{1}_4 & -\mathbb{1}_4 \end{pmatrix} \begin{pmatrix} \Xi_+ & 0 \\ 0 & \Xi_- \end{pmatrix} \begin{pmatrix} \mathbb{1}_4 & \mathbb{1}_4 \\ \mathbb{1}_4 & -\mathbb{1}_4 \end{pmatrix} \quad (75)$$

where $\mathbb{1}_4$ are 4×4 identity matrices, and

$$\Xi_{\pm} = \cos(2\tau) \mathbf{A}_1 \pm i \sin(2\tau) \mathbf{A}_2. \quad (76)$$

Let us denote the Jordan decomposition of Ξ_{\pm} by

$$\Xi_{\pm} = \mathbf{P}_{\pm} \mathbf{D}_{\pm} \mathbf{P}_{\pm}^{-1}, \quad (77)$$

where \mathbf{P}_{\pm} are invertible matrices and \mathbf{D}_{\pm} are the Jordan blocks. It follows from equations (75) and (77) that

$$\begin{aligned} \mathbf{B}^n &= \frac{1}{2} \begin{pmatrix} \mathbf{P}_+ & \mathbf{P}_- \\ \mathbf{P}_+ & -\mathbf{P}_- \end{pmatrix} \begin{pmatrix} \mathbf{D}_+^n & 0 \\ 0 & \mathbf{D}_-^n \end{pmatrix} \begin{pmatrix} \mathbf{P}_+^{-1} & \mathbf{P}_+^{-1} \\ \mathbf{P}_-^{-1} & -\mathbf{P}_-^{-1} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \mathbf{P}_+ \mathbf{D}_+^n \mathbf{P}_+^{-1} + \mathbf{P}_- \mathbf{D}_+^n \mathbf{P}_-^{-1} & \mathbf{P}_+ \mathbf{D}_+^n \mathbf{P}_+^{-1} - \mathbf{P}_- \mathbf{D}_+^n \mathbf{P}_-^{-1} \\ \mathbf{P}_+ \mathbf{D}_-^n \mathbf{P}_+^{-1} - \mathbf{P}_- \mathbf{D}_-^n \mathbf{P}_-^{-1} & \mathbf{P}_+ \mathbf{D}_-^n \mathbf{P}_+^{-1} + \mathbf{P}_- \mathbf{D}_-^n \mathbf{P}_-^{-1} \end{pmatrix}. \end{aligned} \quad (78)$$

In view of equation (74), we use equation (78) to get

$$\mathbf{B}_{0,m+4} = \frac{1}{2} [\mathbf{P}_+ \mathbf{D}_+^n \mathbf{P}_+^{-1}]_{0m} - \frac{1}{2} [\mathbf{P}_- \mathbf{D}_-^n \mathbf{P}_-^{-1}]_{0m} \quad (79)$$

and

$$B_{4,m+4} = \frac{1}{2} [P_+ D_+^n P_+^{-1}]_{0m} + \frac{1}{2} [P_- D_-^n P_-^{-1}]_{0m}. \tag{80}$$

Substituting the above into equation (74) leads to

$$D(n) = \sum_{m=0}^3 [P_- D_-^n P_-^{-1}]_{0m} \rho_m = \sum_{m=0}^3 [\Xi_-^n]_{0m} \rho_m, \tag{81}$$

which gives precisely equation (20). This concludes the proof of Proposition 2. \square

6.4. Derivation of equations (33) and (40)

In this section, we derive limiting cases of the decoherence function (81), assuming each ancilla is initialized in the state $\rho_A = |+\rangle\langle +|_A$, (22). Since for this state $\rho_1 = 1$ and $\rho_2 = \rho_3 = 0$, from (57) we have

$$\begin{aligned} & \cos(2\tau) A_1 - i \sin(2\tau) A_2 \\ &= \begin{pmatrix} \cos(2\tau) & -i \sin(2\tau) \cos(2\varepsilon) & 0 & 0 \\ \cos(2\tau) \cos(2\varepsilon) & -i \sin(2\tau) & 0 & 0 \\ 0 & 0 & 0 & -i \cos(2\tau) \sin(2\varepsilon) \\ 0 & 0 & -\sin(2\tau) \sin(2\varepsilon) & 0 \end{pmatrix}, \end{aligned} \tag{82}$$

such that combining (81) and (82) yields

$$D(n) = [C^n]_{00} + [C^n]_{01}, \quad C = \begin{pmatrix} \cos(2\tau) & -i \sin(2\tau) \cos(2\varepsilon) \\ \cos(2\tau) \cos(2\varepsilon) & -i \sin(2\tau) \end{pmatrix}. \tag{83}$$

The matrix C can be diagonalized as

$$C = P \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix} P^{-1}, \tag{84}$$

where the eigenvalues are $z_{\pm} = \frac{1}{2}(e^{-2i\tau} \pm \Delta)$, and $\Delta = \sqrt{\cos(4\tau) - i \cos(4\varepsilon) \sin(4\tau)}$. Furthermore, the matrix P and its inverse P^{-1} read

$$P = \begin{pmatrix} q_+ & q_- \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{\kappa} \begin{pmatrix} 1 & -q_- \\ -1 & q_+ \end{pmatrix}, \tag{85}$$

with

$$q_{\pm}(\varepsilon, \tau) = \frac{\kappa}{2\Delta} (e^{2i\tau} \pm \Delta), \quad \kappa(\varepsilon, \tau) = q_+ - q_-. \tag{86}$$

Let us for now assume that $\varepsilon, \tau \neq \frac{\pi}{4}$, implying $\cos(2\varepsilon) \cos(2\tau) \neq 0$. We can then proceed to calculate powers of C using (84). In doing so, the decoherence function (83) becomes $D(n) = \frac{1}{\kappa}(q_+(1 - q_-)z_+^n - q_-(1 - q_+)z_-^n)$, which can be rewritten as

$$D(n) = \zeta_+(\varepsilon, \tau) \left(\frac{e^{-2i\tau} + \Delta}{2} \right)^n - \zeta_-(\varepsilon, \tau) \left(\frac{e^{-2i\tau} - \Delta}{2} \right)^n, \tag{87}$$

where

$$\zeta_{\pm}(\varepsilon, \tau) = \frac{1}{\Delta} \left(1 - \frac{e^{2i\tau} \mp \Delta}{2 \cos(2\varepsilon) \cos(2\tau)} \right) \frac{e^{2i\tau} \pm \Delta}{2}. \tag{88}$$

Note that since $|\Delta|^4 = \cos^2(4\tau) + \sin^2(4\tau) \cos^2(4\varepsilon) \leq 1$, with equality if and only if $\cos(4\varepsilon) = \pm 1$, we have $|\Delta| < 1$ for $\varepsilon \neq \frac{\pi}{4}$, and therefore $\frac{1}{2}|e^{-2i\tau} \pm \Delta| < 1$. It follows from (87) that $|D(n)| \leq Ce^{-\gamma n}$ for some $C, \gamma > 0$. This implies initial coherences of S to decay exponentially in the collision number n whenever $\varepsilon \neq \frac{\pi}{4}$.

Next we consider the following two limits of (87), which are well defined given that $D(n)$ is a continuous function of both ε and τ :

- (i) As $\varepsilon \rightarrow \frac{\pi}{4}$, we obtain $\zeta_+(\pi/4, \tau) = 1$ and $\zeta_-(\pi/4, \tau) = 0$, giving

$$\lim_{\varepsilon \rightarrow \pi/4} D(n) = (\cos(2\tau))^n. \tag{89}$$

(ii) As $\tau \rightarrow \frac{\pi}{4}$, we obtain for $n \geq 1$,

$$\lim_{\tau \rightarrow \pi/4} \zeta_+(\varepsilon, \tau) \left(\frac{e^{-2i\tau} + \Delta}{2} \right)^n = 0, \quad \lim_{\tau \rightarrow \pi/4} \zeta_-(\varepsilon, \tau) \left(\frac{e^{-2i\tau} - \Delta}{2} \right)^n = -(-i)^n \cos(2\varepsilon),$$

such that

$$\lim_{\tau \rightarrow \pi/4} D(n) = e^{-i\pi n/2} \cos(2\varepsilon) \quad n \geq 1. \tag{90}$$

Hence, we recover the two expressions (33) and (40).

6.5. Derivation of (42)

The decoherence function $D'(n)$ can be derived following the same procedure used to obtain $D(n)$ in section 6.3, where \mathcal{M}' can be similarly represented as a 16×16 matrix. For the sake of brevity, we omit the full derivation and only present the intermediate results needed to construct the solution for $D'(n)$. The matrix representation of (41) in the basis (50) is found to be

$$M' = \left(\begin{array}{cc|cc} A'_1 & A'_2 & B_1 & B_2 \\ A'_2 & A'_1 & B_2 & B_1 \end{array} \right), \tag{91}$$

where

$$A'_1 = A_1, \quad A'_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i\rho_2 \sin(2\varepsilon) & 0 \\ 0 & 0 & -i\rho_1 \sin(2\varepsilon) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{92}$$

and

$$B_1 = \begin{pmatrix} \cos(2\tau) & 0 & 0 & 0 \\ \rho_1 \cos(2\varepsilon) \cos(2\tau) & 0 & 0 & -i\rho_2 \sin(2\varepsilon) \\ \rho_2 \cos(2\varepsilon) \cos(2\tau) & 0 & 0 & -i\rho_1 \sin(2\varepsilon) \\ \rho_3 \cos(2\tau) & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \rho_1 \cos(2\varepsilon) & 0 & 0 \\ 0 & \rho_2 \cos(2\varepsilon) & 0 & 0 \\ 0 & \rho_3 & 0 & 0 \end{pmatrix}. \tag{93}$$

By again following the same steps outlined in the proof of Proposition 1, we obtain

$$D'(n) = \sum_{m=0}^3 [(\mathbf{B}_1 - i \sin(2\tau)\mathbf{B}_2)^m]_{0m} \rho_m, \tag{94}$$

which depends only on the matrices $\mathbf{B}_{1,2}$. To further simplify this expression one can set $\rho_1 = 1$, $\rho_2 = \rho_3 = 0$, corresponding to when each ancilla is initialized in the state $\rho_A = |+\rangle\langle +|_A$, (22). We then get $(D'(0) = 1)$

$$D'(n) = e^{-2i\tau} (\cos(2\tau) - i \sin(2\tau) \cos(2\varepsilon))^{n-1}, \quad n \geq 1, \tag{95}$$

giving (42).

Data availability statement

No new data were created or analysed in this study.

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